Detection of Scalar Gravitational Waves

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Abstract. In this talk I review recent progresses in the detection of scalar gravitational waves. Furthermore, in the framework of the Jordan-Brans-Dicke theory, I compute the signal to noise ratio for a resonant mass detector of spherical shape and for binary sources and collapsing stars. Finally I compare these results with those obtained from laser interferometers and from Einsteinian gravity.

1. Introduction

The efforts aimed at the detection of gravitational waves (GW) started more than a quarter of century ago and have been, so far, unsuccessful [1, 2]. Resonant bars have proved their reliability, being capable of continous data gathering for long periods of time [3, 4]. Their energy sensitivity has improved of more than four orders of magnitude since Weber's pioneering experiment. But a further improvement is still necessary to achieve successful detection. While further developments of bar detectors are under way, two new generations of earth based experiments have been proposed: detectors based on large laser interferometers are already under construction [5], resonant detectors of spherical shape are under study [2].

In this lecture I report on a series of papers [6] in which the opportunity of introducing resonant mass detectors of spherical shape was studied. As a general motivation for their study, spherical detectors have the advantage over bar-shaped detectors of a larger degree of symmetry. This translates into the possibility of building detectors of greater mass and consequently of higher cross section.

Besides this obvious observation, the higher degree of symmetry enjoyed by the spherical shape puts such a detector in the unique position of being able of detecting GW's with a spin content different from 2. This means to test non-Einsteinian theories of gravity.

I would in fact now like to remind the reader of the very special position of Einstein's general relativity (GR) among the possible gravitational theories. Theories of gravitation, in fact, can be divided into two families: metric and non-metric theories [7]. The former can be defined to be all theories obeying the following three postulates:

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- spacetime is endowed with a metric;
- the world lines of test particles are geodesic of the above mentioned metric;
- in local free-falling frames, the non-gravitational laws of physics are those of special relativity.

It is an obvious consequence of these postulates that a metric theory obeys the principle of equivalence. More succintly a theory is said to be metric if the action of gravitation on the matter sector is due exclusively to the metric tensor. GR is the most famous example of a metric theory. Kaluza-Klein type theories, also belong to this class along with the Brans-Dicke theory. Different representatives of this class differ for their equations of motion which in turn can be deduced from a lagrangian principle. Since there seems to be no compelling experimental or theoretical reasons to introduce non-Einsteinian or non-metric theories, they are sometimes considered a curiosity. This point should perhaps be reconsidered. String theories are in fact the most serious candidate for a theory of quantum gravity, the standard cosmological model has been emended with the introduction of inflation and even the introduction of a cosmological constant (which seems to be needed to explain recent cosmological data) could imply the existence of other gravitationally coupled fields. In all of the above cited cases I am forced to introduce fields which are non-metrically coupled in the sense explained above.

In the first section of this lecture I will explain that a spherical detector is able to detect any spin component of an impinging GW. Moreover its vibrational eigenvalues can be divided into two sets called spheroidal and toroidal. Only the first set couples to the metric. This leads to the opportunity of using such a detector as a veto for non-Einsteinian theories. In the second section I take as a model the Jordan-Brans-Dicke (JBD), in which along with the metric I also have a scalar field which is metrically coupled. I am then able to study the signal to noise ratio for sources such as binary systems and collapsing stars and compare the strenght of the scalar signal with respect to the tensor one. Finally in the third and last section I repeat this computation in the case of the hollow sphere which seems to be the detector which is most likely to be built.

2. Testing Theories of Gravity

2.1. Free Vibrations of an Elastic Sphere

Before discussing the interaction with an external GW field, let us consider the basic equations governing the free vibrations of a perfectly homogeneous, isotropic sphere of radius R, made of a material having density ρ and Lamé coefficients λ and μ [8].

Following the notation of [9], let $x_i, i = 1, 2, 3$ be the equilibrium position of the element of the elastic sphere and x'_i be the deformed position then $u_i = x'_i - x_i$ is the displacement vector. Such vector is assumed small, so that the linear theory of elasticity is applicable. The strain tensor is defined as $u_{ij} = (1/2)(u_{i,j} + u_{j,i})$ and is related to the stress tensor by $\sigma_{ij} = \delta_{ij}\lambda u_{ll} + 2\mu u_{ij}$. The equations of motion of the free vibrating sphere are thus [8]

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x^j}(\delta_{ij}\lambda u_{ll} + 2\mu u_{ij}) \tag{2.1}$$

with the boundary condition:

$$n_j \sigma_{ij} = 0 \tag{2.2}$$

at $r = R$ where $n_i \equiv x_i/r$ is the unit normal. These conditions simply state that the surface of the sphere is free to vibrate. The displacement u_i is a time-dependent vector, whose time dependence can be factorised as $u_i(\vec{x}, t) = u_i(\vec{x}) exp(i\omega t)$, where ω is the frequency. The equations of motion then become:

$$\mu \nabla^2 u_i + (\lambda + \mu)\nabla_i(\nabla_j u_j) = -\omega^2 \rho u_i \tag{2.3}$$

Their solutions can be expressed as a sum of a longitudinal and two transverse vectors [10]:

$$\vec{u}(\vec{x}) = C_0 \vec{\nabla}\phi(\vec{x}) + C_1 \vec{L}\chi(\vec{x}) + C_2 \vec{\nabla} \times \vec{L}\chi(\vec{x}) \tag{2.4}$$

where C_0, C_1, C_2 are dimensioned constants and $\vec{L} \equiv \vec{x} \times \vec{\nabla}$ is the angular momentum operator. Regularity at $r = 0$ restricts the scalar functions ϕ and χ to be expressed as $\phi(r, \theta, \varphi) \equiv j_l(qr)Y_{lm}(\theta, \varphi)$ and $\chi(r, \theta, \varphi) \equiv j_l(kr)Y_{lm}(\theta, \varphi)$. $Y_{lm}(\theta, \varphi)$ are the spherical harmonics and j_l the spherical Bessel functions [11]:

$$j_l(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right) \tag{2.5}$$

$q^2 \equiv \rho\omega^2/(\lambda + 2\mu)$ and $k^2 \equiv \rho\omega^2/\mu$ are the longitudinal and transverse wave vectors respectively.

Imposing the boundary conditions (2.2) at $r = R$ yields two families of solutions:

- *Toroidal* modes: these are obtained by setting $C_0 = C_2 = 0$, and $C_1 \neq 0$. In this case the displacements in (2.4) can be written in terms of the basis:

  $$\vec{\psi}^T_{nlm}(r, \theta, \varphi) = T_{nl}(r)\vec{L}Y_{lm}(\theta, \varphi) \tag{2.6}$$

  with $T_{nl}(r)$ proportional to $j_l(k_{nl}r)$. The eigenfrequencies are determined by the boundary conditions (2.2) which read [10]

  $$f_1(kR) = 0 \tag{2.7}$$

  where

  $$f_1(z) \equiv \frac{d}{dz}\left[\frac{j_l(z)}{z}\right]. \tag{2.8}$$

- *Spheroidal* modes: these are obtained by setting $C_1 = 0$, $C_0 \neq 0$ and $C_2 \neq 0$. The displacements of (2.4) can be expanded in the basis

  $$\vec{\psi}^S_{nlm}(\vec{x}) = A_{nl}(r)Y_{lm}(\theta, \varphi)\vec{n} - B_{nl}(r)\vec{n} \times \vec{L}Y_{lm}(\theta, \varphi) \tag{2.9}$$

  where $A_{nl}(r)$ and $B_{nl}(r)$ are dimensionless radial eigenfunctions [9], which can be expressed in terms of the spherical Bessel functions and their derivatives. The eigenfrequencies are determined by the boundary conditions (2.2) which read [9]

  $$det \begin{pmatrix} f_2(qR) - \frac{\lambda}{2\mu}q^2R^2 f_0(qR) & l(l+1)f_1(kR) \\ f_1(qR) & \frac{1}{2}f_2(kR) + [\frac{l(l+1)}{2} - 1]f_0(kR) \end{pmatrix} = 0 \tag{2.10}$$

  where

  $$f_0(z) \equiv \frac{j_l(z)}{z^2} \quad f_2(z) \equiv \frac{d^2}{dz^2}j_l(z) \tag{2.11}$$

The eigenfrequencies can be determined numerically for both toroidal and spheroidal vibrations. Each mode of order l is $(2l+1)$-fold degenerate. The eigenfrequency values can be obtained from :

$$\omega_{nl} = \sqrt{\frac{\mu}{\rho}}\frac{(kR)_{nl}}{R} \tag{2.12}$$

2.2. Interaction of a Metric GW with the Sphere Vibrational Modes

The detector is assumed to be non-relativistic (with sound velocity $v_s \ll c$ and radius $R \ll \lambda$ the GW wavelength) and endowed with a high quality factor ($Q_{nl} = \omega_{nl}\tau_{nl} \gg 1$, where τ_{nl} is the decay time of the mode nl). The displacement $\vec{u}$ of a point in the detector can be decomposed in normal modes as:

$$\vec{u}(\vec{x},t) = \sum_N A_N(t)\vec{\psi}_N(\vec{x}) \tag{2.13}$$

where N collectively denotes the set of quantum numbers identifying the mode. The basic equation governing the response of the detector is [12]

$$\ddot{A}_N(t) + \tau_N^{-1}\dot{A}_N(t) + \omega_N^2 A_N(t) = f_N(t) \tag{2.14}$$

I assume that the gravitational interaction obeys the principle of equivalence which has been experimentally supported to high accuracy. In terms of the so-called electric components of the Riemann tensor $E_{ij} \equiv R_{0i0j}$, the driving force $f_N(t)$ is then given by [13]

$$f_N(t) = -M^{-1}E_{ij}(t)\int \psi_N^{i*}(\vec{x})x^j\rho d^3x \tag{2.15}$$

where M is the sphere mass and I consider the density ρ as a constant. In any metric theory of gravity E_{ij} is a 3×3 symmetric tensor, which depends on time, but not on spatial coordinates.

Let us now investigate which sphere eigenmodes can be excited by a metric GW, *i.e.* which sets of quantum numbers N give a non-zero driving force.

a) *Toroidal* modes

The eigenmode vector, ψ^T_{nlm} can be expressed as in eq. (2.6). Up to an adimensional normalisation constant C, the driving force is

$$\begin{aligned} f_N^{(T)}(t) \;\; = \;\; & -e^{-i\omega_N t}\frac{3C}{4\pi R^3}\int_0^R drr^3 j_l(k_{nl}^{(T)}r)\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi \\ & \Big\{\frac{E_{yy}-E_{xx}}{2}\Big(\sin\theta\sin 2\phi\frac{\partial Y^*_{lm}}{\partial\theta} + \cos\theta\cos 2\phi\frac{\partial Y^*_{lm}}{\partial\phi}\Big) \\ & + E_{xy}\Big(\sin\theta\cos 2\phi\frac{\partial Y^*_{lm}}{\partial\theta} - \cos\theta\sin 2\phi\frac{\partial Y^*_{lm}}{\partial\phi}\Big) \\ & + E_{xz}\Big[-\sin\phi\cos\theta\frac{\partial Y^*_{lm}}{\partial\theta} + (\sin\theta\cos\phi - \frac{\cos^2\theta}{\sin\theta}\cos\phi)\frac{\partial Y^*_{lm}}{\partial\phi}\Big] \\ & + E_{yz}\Big[\cos\phi\cos\theta\frac{\partial Y^*_{lm}}{\partial\theta} + (\sin\theta\sin\phi - \frac{\cos^2\theta}{\sin\theta}\sin\phi)\frac{\partial Y^*_{lm}}{\partial\phi}\Big] \\ & + \Big(E_{zz} - \frac{E_{xx}+E_{yy}}{2}\Big)\cos\theta\frac{\partial Y^*_{lm}}{\partial\phi}\Big\} \end{aligned} \tag{2.16}$$

Using the equations

$$\frac{\partial Y_{lm}^*}{\partial \theta} = (-)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} \frac{\partial P_l^m(\cos\theta)}{\partial \theta} e^{-im\phi} \tag{2.17}$$

and

$$\frac{\partial Y_{lm}^*}{\partial \phi} = -im(-)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} P_l^m(\cos\theta) e^{-im\phi} \tag{2.18}$$

the integration over ϕ can be performed. Eq. (2.16) then contains integrals over θ of the form:

$$\int_0^\pi \left[(\sin^2\theta - \cos^2\theta) P_l^{\pm 1}(\cos\theta) - \sin\theta\cos\theta \frac{\partial P_l^{\pm 1}(\cos\theta)}{\partial\theta}\right] d\theta \tag{2.19}$$

and

$$\int_0^\pi \left[2\sin\theta\cos\theta P_l^{\pm 2}(\cos\theta) + \sin^2\theta \frac{\partial P_l^{\pm 2}(\cos\theta)}{\partial\theta}\right] d\theta \tag{2.20}$$

After integration by parts, the derivative terms in eqs. (2.19) and (2.20) exactly cancel the non-derivative ones. The remaining boundary terms vanish too, thanks to the periodicity of the trigonometric functions and to the regularity of the associated Legendre polynomials. The vanishing of the above integrals has a profound physical consequence. It means that in any metric theory of gravity the toroidal modes of the sphere cannot be excited by GW and can thus be used as a veto in the detection.

b) *Spheroidal* modes

The forcing term is given by:

$$f_N^{(S)}(t) = -M^{-1} E_{ij}(t) \int x^j \Big(\frac{x^i}{r} A_N(r) Y_{lm}(\theta,\varphi) - B_N(r) \epsilon^{ink} \frac{x_n}{r} L_k Y_{lm}(\theta,\varphi)\Big) \rho d^3x \tag{2.21}$$

One is thus lead to compute integrals of the following types

$$\int x^j x^i Y_{lm}(\theta,\varphi) d^3x \tag{2.22}$$

and

$$\int x^j x^i L_k Y_{lm}(\theta,\varphi) d^3x \tag{2.23}$$

Since the product $x^i x^j$ can be expressed in terms of the spherical harmonics with $l = 0, 2$ and the angular momentum operator does not change the value of l, one immediately concludes that in any metric theory of gravity only the $l = 0, 2$ spheroidal modes of the sphere can be excited. At the lowest level there are a total of five plus one independent spheroidal modes that can be used for GW detection and study.

2.3. Measurements of the Sphere Vibrations and Wave Polarization States

From the analysis of the spheroidal modes active for metric GW, I now want to infer the field content of the theory. For this purpose it is convenient to express the Riemann tensor in a null (Newman-Penrose) tetrad basis [7].

To lowest non-trivial order in the perturbation the six indipendent "electric" components of the Riemann tensor may be expressed in terms of the Newmann-Penrose (NP) parameters as

$$E_{ij} = \begin{pmatrix} -Re\Psi_4 - \Phi_{22} & Im\Psi_4 & -2\sqrt{2}Re\Psi_3 \\ Im\Psi_4 & Re\Psi_4 - \Phi_{22} & 2\sqrt{2}Im\Psi_3 \\ -2\sqrt{2}Re\Psi_3 & 2\sqrt{2}Im\Psi_3 & -6\Psi_2 \end{pmatrix} \tag{2.24}$$

The NP parameters allow the identification of the spin content of the metric theory responsible for the generation of the wave [7]. The classification can be summarized in order of incresing complexity as follows:

- General Relativity (spin 2): $\Psi_4 \neq 0$ while $\Psi_2 = \Psi_3 = \Phi_{22} = 0$.
- Tensor-scalar theories (spin 2 and 0): $\Psi_4 \neq 0$, $\Psi_3 = 0$, $\Psi_2 \neq 0$ and/or $\Phi_{22} \neq 0$ (*e.g.* Brans-Dicke theory with $\Psi_4 \neq 0$, $\Psi_2 = 0$, $\Psi_3 = 0$ and $\Phi_{22} \neq 0$).
- Tensor-vector theories (spin 2 and 1): $\Psi_4 \neq 0$, $\Psi_3 \neq 0$, $\Phi_{22} = \Psi_2 = 0$.
- Most General Metric Theory (spin 2, 1 and 0): $\Psi_4 \neq 0$, $\Psi_2 \neq 0$, $\Psi_3 \neq 0$ and $\Phi_{22} \neq 0$, (*e.g.* Kaluza-Klein theories with $\Psi_4 \neq 0$, $\Psi_3 \neq 0$, $\Phi_{22} \neq 0$ while $\Psi_2 = 0$).

In eq. (2.24), I have assumed that the wave comes from a localized source with wave vector $\vec{k}$ parallel to the z axis of the detector frame. In this case the NP parameters (and thus the wave polarisation states) can be uniquely determined by monitoring the six lowest spheroidal modes. If the direction of the incoming wave is not known two more unknowns appear in the problem, *i.e.* the two angles of rotation of the detector frame needed to align $\vec{k}$ along the z axis. In order to dispose of this problem one can envisage the possibility of combining the pieces of information from an array of detectors [14]. I restrict my attention to the simplest case in which the source direction is known.

In order to infer the value of the NP parameters from the measurements of the excited vibrational modes of the sphere, I decompose E_{ij} in terms of spherical harmonics. In fact, the experimental measurements give the vibrational amplitudes of the sphere modes which are also naturally expanded in the above basis. The use of the same basis makes the connection between the NP parameters and the measured amplitudes straightforward. In formulae

$$E_{ij}(t) = \sum_{l,m} c_{l,m}(t) S_{ij}^{(l,m)} \tag{2.25}$$

where $S_{ij}^{(0,0)} \equiv \delta_{ij}/\sqrt{4\pi}$ (with δ_{ij} the Kronecker symbol) and $S_{ij}^{(2,m)}$ $(m = -2, ..2)$ are five linearly independent symmetric and traceless matrices such as

$$S_{ij}^{(l,m)} n^i n^j = Y_{lm}, \quad l = 0, 2 \tag{2.26}$$

The vector n_i in eqs. (2.26) has been defined after eq. (2.2).

Taking the scalar product I find

$$\begin{aligned} c_{0,0}(t) &= \frac{4\pi}{3} S_{ij}^{0,0} E_{ij}(t) \\ c_{2,m}(t) &= \frac{8\pi}{15} S_{ij}^{2,m} E_{ij}(t) \end{aligned} \tag{2.27}$$

and then for the NP parameters

$$\begin{aligned} \Phi_{22} &= \sqrt{\frac{5}{16\pi}} c_{2,0}(t) - \sqrt{\frac{1}{4\pi}} c_{0,0}(t) \quad \Psi_2 = -\frac{1}{12}\sqrt{\frac{5}{\pi}} c_{2,0}(t) - \frac{1}{12}\sqrt{\frac{1}{\pi}} c_{0,0}(t) \\ Re\Psi_4 &= -\sqrt{\frac{15}{32\pi}} [c_{2,2} + c_{2,-2}] \quad Im\Psi_4 = -i\sqrt{\frac{15}{32\pi}} [c_{2,2} - c_{2,-2}] \end{aligned}$$

$$Re\Psi_3 = \frac{1}{16}\sqrt{\frac{15}{\pi}}[c_{2,1} - c_{2,-1}] \quad Im\Psi_3 = \frac{i}{16}\sqrt{\frac{15}{\pi}}[c_{2,1} + c_{2,-1}] \tag{2.28}$$

Eqs. (2.28) relate the measurable quantities $c_{l,m}$ with the GW polarization states, described by the NP parameters. Eq. (2.28) can be put in correspondence with the output of experimental measurements if the $c_{l,m}$ are substituted with their Fourier components at the quadrupole and monopole resonant frequencies which, for the sake of simplicity, I collectively denote by ω_0. The $c_{l,m}(\omega_0)$ can be determined in the following way: once the Fourier amplitudes $A_N(\omega_0)$ are measured, by Fourier transforming (2.14) and (2.15) I get the Riemann amplitudes $E_{ij}(\omega_0)$ which, using (2.27), yield the desired result.

In order to determine the $A_N(\omega_0)$ amplitudes from a given GW signal the following two conditions must be fulfilled:

- the vibrational states of the five-fold degenerate quadrupole and monopole modes must be determined. The quadrupole modes can be studied by properly combining the outputs of a set of at least five motion sensors placed in independent positions on the sphere surface. Explicit formulas for practical and elegant configurations of the motion sensors have been reported by various authors [15, 16]. The vibrational state of the monopole mode is provided directly by the output of any of the above mentioned motion sensors. If resonant motion sensors are used, since the quadrupole and monopole states resonate at different frequencies, a sixth sensor is needed.

- The spectrum of the GW signal must be sufficiently broadband to overlap with the antenna quadrupole and monopole frequencies.

3. Gravitational Wave Radiation in the Jordan-Brans-Dicke Theory

In this section I analyze the signal emitted by a compact binary system in the Jordan-Brans-Dicke theory. I compute the scalar and tensor components of the power radiated by the source and study the scalar waveform. Eventually I consider the detectability of the scalar component of the radiation by interferometers and resonant-mass detectors.

3.1. Scalar and Tensor GWs in the JBD Theory

In the Jordan-Fierz frame, in which the scalar field mixes with the metric but decouples from matter, the action reads [17]

$$\begin{aligned} S &= S_{\rm grav}[\phi, g_{\mu\nu}] + S_{\rm m}[\psi_m, g_{\mu\nu}] \\ &= \frac{c^3}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega_{BD}}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] + \frac{1}{c}\int d^4x L_{\rm m}[\psi_{\rm m}, g_{\mu\nu}] \quad , \end{aligned} \tag{3.1}$$

where ω_{BD} is a dimensionless constant, whose lower bound is fixed to be $\omega_{BD} \approx 600$ by experimental data [18], $g_{\mu\nu}$ is the metric tensor, ϕ is a scalar field, and $\psi_{\rm m}$ collectively denotes the matter fields of the theory.

As a preliminary analysis, I perform a weak field approximation around the background given by a Minkowskian metric and a constant expectation value for the scalar field

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ \varphi &= \varphi_0 + \xi \quad . \end{aligned} \tag{3.2}$$

The standard parametrization $\varphi_0 = 2(\omega_{BD} + 2)/G(2\omega_{BD} + 3)$, with G the Newton constant, reproduces GR in the limit $\omega_{BD} \to \infty$, which implies $\varphi_0 \to 1/G$. Defining the new field

$$\theta_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h - \eta_{\mu\nu}\frac{\xi}{\varphi_0} \tag{3.3}$$

where h is the trace of the fluctuation $h_{\mu\nu}$, and choosing the gauge

$$\partial_\mu \theta^{\mu\nu} = 0 \tag{3.4}$$

one can write the field equations in the following form

$$\partial_\alpha \partial^\alpha \theta_{\mu\nu} = -\frac{16\pi}{\varphi_0}\, \tau_{\mu\nu} \tag{3.5}$$

$$\partial_\alpha \partial^\alpha \xi = \frac{8\pi}{2\omega_{BD} + 3}\, S \tag{3.6}$$

where

$$\tau_{\mu\nu} = \frac{1}{\varphi_0}(T_{\mu\nu} + t_{\mu\nu}) \tag{3.7}$$

$$S = -\frac{T}{2(2\omega_{BD} + 3)}\left(1 - \frac{1}{2}\theta - 2\frac{\xi}{\varphi_0}\right) - \frac{1}{16\pi}\left[\frac{1}{2}\partial_\alpha(\theta\partial^\alpha\xi) + \frac{2}{\varphi_0}\partial_a(\xi\partial^\alpha\xi)\right] \tag{3.8}$$

In the equation (3.7), $T_{\mu\nu}$ is the matter stress-energy tensor and $t_{\mu\nu}$ is the gravitational stress-energy pseudo-tensor, that is a function of quadratic order in the weak gravitational fields $\theta_{\mu\nu}$ and ξ. The reason why I have written the field equations at the quadratic order in $\theta_{\mu\nu}$ and ξ is that in this way, as I will see later, the expressions for $\theta_{\mu\nu}$ and ξ include all the terms of order $(v/c)^2$, where v is the typical velocity of the source (Newtonian approximation).

Let us now compute τ^{00} and S at the order $(v/c)^2$. Introducing the Newtonian potential U produced by the rest-mass density ρ

$$U(\vec{x}, t) = \int \frac{\rho(\vec{x}', t)}{\mid \vec{x} - \vec{x}' \mid}\, d^3x' \tag{3.9}$$

the total pressure p and the specific energy density Π (that is the ratio of energy density to rest-mass density) I get (for a more detailed derivation, see [7]):

$$\tau^{00} = \frac{1}{\varphi_0}\rho \quad , \tag{3.10}$$

$$\begin{aligned} S &\simeq -\frac{T}{2(2\omega_{BD} + 3)}\left(1 - \frac{1}{2}\theta - 2\frac{\xi}{\varphi_0}\right) \\ &= \frac{\rho}{2(2\omega_{BD} + 3}\left(1 + \Pi - 3\,\frac{p}{\rho} + \frac{2\omega_{BD} + 1}{\omega_{BD} + 2}\, U\right) \end{aligned} \tag{3.11}$$

Far from the source, the equations (3.5) and (3.6) admit wave–like solutions, which are superpositions of terms of the form

$$\begin{aligned} \theta_{\mu\nu}(x) &= A_{\mu\nu}(\vec{x},\omega)\exp(ik^{\alpha}x_{\alpha}) + c.c. \qquad (3.12) \\ \xi(x) &= B(\vec{x},\omega)\exp(ik^{\alpha}x_{\alpha}) + c.c. \qquad (3.13) \end{aligned}$$

Without affecting the gauge condition (3.4), one can impose $h = -2\xi/\varphi_0$ (so that $\theta_{\mu\nu} = h_{\mu\nu}$). Gauging away the superflous components, one can write the amplitude $A_{\mu\nu}$ in terms of the three degrees of freedom corrisponding to states with helicities ± 2 and 0 [19]. For a wave travelling in the z-direction, one thus obtains

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} - b & e_{12} & 0 \\ 0 & e_{12} & -e_{11} - b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (3.14)$$

where $b = B/\varphi_0$.

3.2. Power emitted in GWs

The power emitted by a source in GWs depends on the stress-energy pseudo-tensor $t^{\mu\nu}$ according to the following expression

$$P_{GW} = r^2 \int \Phi d\Omega = r^2 \int < t^{0k} > \ \hat{x}_k \ d\Omega \qquad (3.15)$$

where r is the radius of a sphere which contains the source, Ω is the solid angle, Φ is the energy flux and the symbol $< ... >$ implies an average over a region of size much larger than the wavelength of the GW. At the quadratic order in the weak fields I find

$$< t_{0z} >= -\hat{z}\frac{\varphi_0 c^4}{32\pi}\left[\frac{4(\omega_{BD}+1)}{\varphi_0^2} < (\partial_0\xi)(\partial_0\xi) > + < (\partial_0 h_{\alpha\beta})(\partial_0 h^{\alpha\beta}) > \right]. \qquad (3.16)$$

Substituting (3.12), (3.13) into (3.16), one gets

$$< t_{0z} >= -\hat{z}\frac{\varphi_0 c^4\omega^2}{16\pi}\left[\frac{2(2\omega_{BD}+3)}{\varphi_0^2} \mid B \mid^2 + A^{\alpha\beta *}A_{\alpha\beta} - \frac{1}{2} \mid A^{\alpha}{}_{\alpha} \mid^2\right], \qquad (3.17)$$

and using (3.14)

$$< t_{0z} >= -\hat{z}\frac{\varphi_0 c^4\omega^2}{8\pi}\left[\mid e_{11} \mid^2 + \mid e_{12} \mid^2 + (2\omega_{BD}+3) \mid b \mid^2\right]. \qquad (3.18)$$

From (3.18) I see that the purely scalar contribution, associated to b, and the traceless tensorial contribution, associated to $e_{\mu\nu}$, are completely decoupled and can thus be treated independently.

3.3. Power emitted in scalar GWs

I now rewrite the scalar wave solution (3.13) in the following way

$$\xi(\vec{x},t) = \xi(\vec{x},\omega)e^{-i\omega t} + c.c. \qquad (3.19)$$

In vacuo, the spatial part of the previous solution (3.19) satisfies the Helmholtz equation

$$(\nabla^2 + \omega^2)\xi(\vec{x}, \omega) = 0 \tag{3.20}$$

The solution of (3.20) can be written as

$$\xi(\vec{x}, \omega) = \sum_{jm} X_{jm} h_j^{(1)}(\omega r) Y_{jm}(\theta, \varphi) \tag{3.21}$$

where $h_j^{(1)}(x)$ are the spherical Hankel functions of the first kind, r is the distance of the source from the observer, $Y_{jm}(\theta, \varphi)$ are the scalar spherical harmonics and the coefficients X_{jm} give the amplitudes of the various multipoles which are present in the scalar radiation field. Solving the inhomogeneous wave equation (3.6), I find

$$X_{jm} = 16\pi i\omega \int_V j_l(\omega r') Y^*_{lm}(\theta, \varphi) S(\vec{x}, \omega)\ dV \tag{3.22}$$

where $j_l(x)$ are the spherical Bessel functions and r' is a radial coordinate which assumes its values in the volume V occupied by the source.

Substituting (3.16) in (3.15), considering the expressions (3.19) and (3.21), and averaging over time, one finally obtains

$$P_{scal} = \frac{(2\omega_{BD} + 3)c^4}{8\pi\varphi_0} \sum_{jm} \mid X_{jm} \mid^2 \tag{3.23}$$

To compute the power radiated in scalar GWs, one has to determine the coefficients X_{jm}, defined in (3.22). The detailed calculations can be found in the appendix A of the third reference in [6], while here I only give the final results. Introducing the reduced mass of the binary system $\mu = m_1 m_2/m$ and the gravitational self-energy for the body a (with $a = 1, 2$)

$$\Omega_a = -\frac{1}{2}\int_{V_a} \frac{\rho(\vec{x})\rho(\vec{x}')}{\mid \vec{x} - \vec{x}' \mid}\ d^3x d^3x' \tag{3.24}$$

one can write the Fourier components with frequency $n\omega_0$ in the Newtonian approximation

$$(X_{00})_n = -\frac{16\sqrt{2\pi}}{3} \frac{i\omega_0\varphi_0}{\omega_{BD} + 2} \frac{m\mu}{a}\ nJ_n(ne) \tag{3.25}$$

$$
\begin{aligned}
(X_{1\pm1})_n &= -\sqrt{\frac{2\pi}{3}}\frac{2i{\omega_0}^2\varphi_0}{\omega_{BD}+2}\left(\frac{\Omega_2}{m_2}-\frac{\Omega_1}{m_1}\right)\ \mu a \\
&\quad \left[\pm J_n'(ne)-\frac{1}{e}(1-e^2)^{1/2}J_n(ne)\right] \qquad (3.26)\\
(X_{20})_n &= \frac{2}{3}\sqrt{\frac{\pi}{5}}\frac{i{\omega_0}^3\varphi_0}{\omega_{BD}+2}\mu a^2 nJ_n(ne) \qquad (3.27)\\
(X_{2\pm2})_n &= \mp 2\sqrt{\frac{\pi}{30}}\frac{i{\omega_0}^3\varphi_0}{\omega_{BD}+2}\mu a^2 \\
&\quad \frac{1}{n}\{(e^2-2)J_n(ne)/(ne^2)+2(1-e^2)J_n'(ne)/e \\
&\mp 2(1-e^2)^{1/2}[(1-e^2)J_n(ne)/e^2-J_n'(ne)/(ne)]\} \qquad (3.28)
\end{aligned}
$$

Substituting these expressions in (3.23), leads to the power radiated in scalar GWs in the n-th harmonic

$$
(P_{scal})_n = P_n^{j=0}+P_n^{j=1}+P_n^{j=2} \qquad (3.29)
$$

where the monopole, dipole and quadrupole terms are respectively

$$
\begin{aligned}
P_n^{j=0} &= \frac{64}{9(\omega_{BD}+2)}\frac{m^3\mu^2G^4}{a^5c^5}n^2J_n^2(ne) \\
&= \frac{64}{9(\omega_{BD}+2)}\frac{m^3\mu^2G^4}{a^5c^5}\ m(n;e) \qquad (3.30)\\
P_n^{j=1} &= \frac{4}{3(\omega_{BD}+2)}\frac{m^2\mu^2G^3}{a^4c^3}\left(\frac{\Omega_2}{m_2}-\frac{\Omega_1}{m_1}\right)^2 \\
&\quad n^2\left[J_n'^2(ne)+\frac{1}{e^2}(1-e^2)J_n^2(ne)\right] \\
&= \frac{4}{3(\omega_{BD}+2)}\frac{m^2\mu^2G^3}{a^4c^3}\left(\frac{\Omega_2}{m_2}-\frac{\Omega_1}{m_1}\right)^2 d(n;e) \qquad (3.31)\\
P_n^{j=2} &= \frac{8}{15(\omega_{BD}+2)}\frac{m^3\mu^2G^4}{a^5c^5}\ g(n;e) \qquad (3.32)
\end{aligned}
$$

The total power radiated in scalar GWs by a binary system is the sum of three terms

$$
P_{scal} = P^{j=0}+P^{j=1}+P^{j=2} \qquad (3.33)
$$

where

$$
\begin{aligned}
P^{j=0} &= \frac{16}{9(\omega_{BD}+2)}\frac{G^4}{c^5}\frac{m_1^2m_2^2m}{a^5}\frac{e^2}{(1-e^2)^{7/2}}\left(1+\frac{e^2}{4}\right) \qquad (3.34)\\
P^{j=1} &= \frac{2}{\omega_{BD}+2}\left(\frac{\Omega_2}{m_2}-\frac{\Omega_1}{m_1}\right)^2\frac{G^3}{c^3}\frac{m_1^2m_2^2}{a^4}\frac{1}{(1-e^2)^{5/2}}\left(1+\frac{e^2}{2}\right) \qquad (3.35)\\
P^{j=2} &= \frac{8}{15(\omega_{BD}+2)}\frac{G^4}{c^5}\frac{m_1^2m_2^2m}{a^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \qquad (3.36)
\end{aligned}
$$

Note that $P^{j=0}, P^{j=1}, P^{j=2}$ all go to zero in the limit $\omega_{BD}\to\infty$.

3.4. Scalar GWs

I now give the explicit form of the scalar GWs radiated by a binary system. To this end, note that the major semi-axis, a, is related to the total energy, E, of the system through the following equation

$$a = -\frac{Gm_1 m_2}{2E} \tag{3.37}$$

Let us consider the case of a circular orbit, remembering that in the last phase of evolution of a binary system this condition is usually satisfied. Furthermore I will also assume $m_1 = m_2$. With these positions only the quadrupole term, (3.32), of the gravitational radiation is different from zero. The total power radiated in GWs, averaged over time, is then given by (3.34)-(3.36)

$$P = \frac{8}{15(\omega_{BD} + 2)} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 m}{d^5} [6(2\omega_{BD} + 3) + 1] \tag{3.38}$$

where d is the relative distance between the two stars. The time variation of d in one orbital period is

$$\dot{d} = -\frac{Gm_1 m_2}{2E^2} P \tag{3.39}$$

Finally, substituting (3.37), (3.38) in (3.39) and integrating over time, one obtains

$$d = 2\left(\frac{2}{15} \frac{12\omega_{BD} + 19}{\omega_{BD} + 2} \frac{G^3 m_1 m_2 m}{c^5}\right)^{1/4} \tau^4 \tag{3.40}$$

where I have defined $\tau = t_c - t$, t_c being the time of the collapse between the two bodies.

From (3.21), (3.25)-(3.28) and one can deduce the form of the scalar field (see appendix B of the third in [6] for details) which, for equal masses, is

$$\xi(t) = -\frac{2\mu}{r(2\omega_{BD} + 3)} \left[v^2 + \frac{m}{d} - (\hat{n} \cdot \vec{v})^2 + \frac{m}{d^3}(\hat{n} \cdot \vec{d})\right] \tag{3.41}$$

where r is the distance of the source from the observer, and $\hat{n}$ is the versor of the line of sight from the observer to the binary system center of mass. Indicating with γ the inclination angle, that is the angle between the orbital plane and the reference plane (defined to be a plane perpendicular to the line of sight), and with ψ the true anomaly, that is the angle between d and the x-axis in the orbital plane x-y, yields $\hat{n} \cdot \vec{d} = d \sin\gamma \sin\psi$. Then from (3.41) one obtains

$$\xi(t) = \frac{2G\mu m}{(2\omega_{BD} + 3)c^4 dr} \sin^2\gamma \cos(2\psi(t)) \tag{3.42}$$

which can also be written as

$$\xi(\tau) = \xi_0(\tau) \sin(\chi(\tau) + \bar{\chi}) \tag{3.43}$$

where $\bar{\chi}$ is an arbitrary phase and the amplitude $\xi_0(\tau)$ is given by

$$\begin{aligned} \xi_0(\tau) &= \frac{2G\mu m}{(2\omega_{BD} + 3)c^4 dr} \sin^2\gamma \\ &= \frac{1}{2(2\omega_{BD} + 3)r} \left(\frac{\omega_{BD} + 2}{12\omega_{BD} + 19}\right)^{1/4} \left(\frac{15G}{2c^{11}}\right)^{1/4} \frac{M_c^{5/4}}{\tau^{1/4}} \sin^2\gamma \end{aligned} \tag{3.44}$$

In the last expression, I have introduced the definition of the chirp mass $M_c = (m_1 m_2)^{3/5}/m^{1/5}$.

Let me now study the interaction of the scalar GWs a spherical GW detector.

As usual, I characterize the sensitivity of the detector by the spectral density of strain $S_h(f)$ $[\mathrm{Hz}]^{-1}$. The optimum performance of a detector is obtained by filtering the output with a filter matched to the signal. The energy signal-to-noise ratio SNR of the filter output is given by the well-known formula:

$$SNR = \int_{-\infty}^{+\infty} \frac{|H(f)|^2}{S_h(f)}\, df \tag{3.45}$$

where $H(f)$ is the Fourier transform of the scalar gravitational waveform $h_s(t) = G\xi_0(t)$.

I must now take into account the astrophysical restrictions on the validity of the waveform (3.43) which is obtained in the Newtonian approximation for point-like masses. In the following, I will take the point of view that this approximation breaks down when there are five cycles remaining to collapse [20, 21].

The five-cycles limit will be used to restrict the range of M_c over which my analysis will be performed. From (3.40), one can obtain

$$\begin{aligned} \omega_g(\tau) &= 2\omega_0 = 2\sqrt{\frac{Gm}{d^3}} \\ &= 2\left(\frac{15c^5}{64G^{5/3}}\right)^{3/8}\left(\frac{\omega_{BD}+2}{12\omega_{BD}+19}\right)^{3/8}\frac{1}{{M_c}^{5/8}}\tau^{3/8} \end{aligned} \tag{3.46}$$

Integrating (3.46) yields the amount of phase until coalescence

$$\chi(\tau) = \frac{16}{5}\left(\frac{15c^5}{64G^{5/3}}\right)^{3/8}\left(\frac{\omega_{BD}+2}{12\omega_{BD}+19}\right)^{3/8}\left(\frac{\tau}{M_c}\right)^{5/8} \tag{3.47}$$

Setting (3.47) equal to the limit period, $T_{5\ cycles} = 5(2\pi)$, solving for τ and using (3.46) leads to

$$\omega_{5\ cycles} = 2\pi(6870\ \mathrm{Hz})\left(\frac{\omega_{BD}+2}{12\omega_{BD}+19}\right)^{3/5}\frac{M_\odot}{M_c} \tag{3.48}$$

Taking $\omega_{BD} = 600$, the previous limit reads

$$\omega_{5\ cycles} = 2\pi(1547\ \mathrm{Hz})\frac{M_\odot}{M_c} \tag{3.49}$$

A GW excites those vibrational modes of a resonant body having the proper simmetry. In the framework of the JBD theory the spheroidal modes with $l = 2$ and $l = 0$ are sensitive to the incoming GW. Thanks to its multimode nature, a single sphere is capable of detecting GW's from all directions and polarizations. I now evaluate the SNR of a resonant-mass detector of spherical shape for its quadrupole mode with $m = 0$ and its monopole mode. In a resonant-mass detector, $S_h(f)$ is a resonant curve and can be characterized by its value at resonance $S_h(f_n)$ and by its half height width [22]. $S_h(f_n)$ can thus be written as

$$S_h(f_n) = \frac{G}{c^3}\frac{4kT}{\sigma_n Q_n f_n} \tag{3.50}$$

Here σ_n is the cross-section associated with the n-th resonant mode, T is the thermodynamic temperature of the detector and Q_n is the quality factor of the mode.

The half height width of $S_h(f)$ gives the bandwidth of the resonant mode

$$\Delta f_n = \frac{f_n}{Q_n} \Gamma_n^{-1/2} \tag{3.51}$$

Here, Γ_n is the ratio of the wideband noise in the n-th resonance bandwidth to the narrowband noise.

From the resonant-mass detector viewpoint, the chirp signal can be treated as a transient GW, depositing energy in a time-scale short with respect to the detector damping time. I can then consider constant the Fourier transform of the waveform within the band of the detector and write [22]

$$SNR = \frac{2\pi \Delta f_n |H(f_n)|^2}{S_h(f_n)} \tag{3.52}$$

The cross-sections associated to the vibrational modes with $l = 0$ and $l = 2$, $m = 0$ are respectively [6]

$$\sigma_{(n0)} = H_n \frac{G M {v_s}^2}{c^3(\omega_{BD} + 2)} \tag{3.53}$$

$$\sigma_{(n2)} = \frac{F_n}{6} \frac{G M {v_s}^2}{c^3(\omega_{BD} + 2)} \tag{3.54}$$

All parameters entering the previous equation refer to the detector M is its mass, v_s the sound velocity and the constants H_n and F_n are given in [6]. The signal-to noise ratio can be calculated analytically by approximating the waveform with a truncated Taylor expansion around $t = 0$, where $\omega_g(t = 0) = \omega_{nl}$ [23, 20]

$$h_s(t) \approx G\xi_0(t = 0) \sin\left[\omega_{nl} t + \frac{1}{2}\left(\frac{d\omega}{dt}\right)_{t=0} t^2\right] \tag{3.55}$$

Using quantum limited readout systems, one finally obtains

$$(SNR_n)_{l=0} = \frac{5 \cdot 2^{1/3} H_n G^{5/3}}{32(\omega_{BD} + 2)(12\omega_{BD} + 19)\hbar c^3} \frac{{M_c}^{5/3} M {v_s}^2}{r^2 {\omega_{n0}}^{4/3}} \sin^4 \gamma \tag{3.56}$$

$$(SNR_n)_{l=2} = \frac{5 \cdot 2^{1/3} F_n G^{5/3}}{192(\omega_{BD} + 2)(12\omega_{BD} + 19)\hbar c^3} \frac{{M_c}^{5/3} M {v_s}^2}{r^2 {\omega_{n0}}^{4/3}} \sin^4 \gamma \tag{3.57}$$

which are respectively the signal-to-noise ratio for the modes with $l = 0$ and $l = 2$, $m = 0$ of a spherical detector.

It has been proposed to realize spherical detectors with 3 meters diameter, made of copper alloys, with mass of the order of 100 tons [24]. This proposed detector has resonant frequencies of $\omega_{12} = 2\pi \cdot 807$ rad/s and $\omega_{10} = 2\pi \cdot 1655$ rad/s. In the case of optimally oriented orbits (inclination angle $\gamma = \pi/2$) and $\omega_{BD} = 600$, the inspiralling of two compact objects of 1.4 solar masses each will then be detected with $SNR = 1$ up to a source distance $r(\omega_{10}) \simeq 30$ kpc and $r(\omega_{12}) \simeq 30$ kpc.

4. The hollow sphere

An appealing variant of the massive sphere is a *hollow* sphere [27]. The latter has the remarkable property that it enables the detector to monitor GW signals in a significantly *lower frequency range* —down to about 200 Hz— than its massive counterpart for comparable sphere masses. This can be considered a positive advantage for a future world wide network of GW detectors, as the sensitivity range of such antenna overlaps with that of the large scale interferometers, now in a rather advanced state of construction [25, 26]. In this Section I study the response of such a detector to the GW energy emitted by a binary system constituted of stars of masses of the order of the solar mass. A hollow sphere obviously has the same symmetry of the massive one, so the general structure of its *normal modes* of vibration is very similar[27] to that of the solid sphere. In particular, the hollow sphere is very well adapted to sense and monitor the presence of scalar modes in the incoming GW signal. The extension of the analysis of the previous Sections to a hollow sphere is quite straightforward and in the following I will only give the main results. Due to the different geometry, the vibrational modes of a hollow sphere differ from those studied in Section 2. In the case of a hollow sphere, I have two boundaries given by the outer and the inner surfaces of the solid itself. I use the notation a for the inner radius, and R for the outer radius. The boundary conditions are thus expressed by

$$\sigma_{ij} n_j = 0 \qquad \text{at} \quad r = R \quad \text{and at} \quad r = a \quad (R \geq a \geq 0), \tag{4.58}$$

(2.3) must now be solved subject to this boundary conditions. The solution that leads to spheroidal modes is still (2.9) where the radial functions $A_{nl}(r)$ and $B_{nl}(r)$ have rather complicated expressions:

$$\begin{aligned} A_{nl}(r) &= C_{nl} \left[\frac{1}{q^S_{nl}} \frac{d}{dr} j_l(q^S_{nl} r) - l(l+1) K_{nl} \frac{j_l(k^S_{nl} r)}{k^S_{nl} r} + \right. \\ &\qquad \left. + D_{nl} \frac{1}{q^S_{nl}} \frac{d}{dr} y_l(q^S_{nl} r) - l(l+1) \tilde{D}_{nl} \frac{y_l(k^S_{nl} r)}{k^S_{nl} r} \right] \end{aligned} \tag{4.59}$$

$$\begin{aligned} B_{nl}(r) &= C_{nl} \left[\frac{j_l(q^S_{nl} r)}{q^S_{nl} r} - K_{nl} \frac{1}{k^S_{nl} r} \frac{d}{dr} \left\{ r\, j_l(k^S_{nl} r) \right\} + \right. \\ &\qquad \left. + D_{nl} \frac{y_l(q^S_{nl} r)}{q^S_{nl} r} - \tilde{D}_{nl} \frac{1}{k^S_{nl} r} \frac{d}{dr} \left\{ r\, y_l(k^S_{nl} r) \right\} \right] \end{aligned} \tag{4.60}$$

Here $k^S_{nl} R$ and $q^S_{nl} R$ are dimensionless *eigenvalues*, and they are the solution to a rather complicated algebraic equation for the frequencies $\omega = \omega_{nl}$ —see [27] for details. In (4.59) and (4.60) I have set

$$K_{nl} \equiv \frac{C_{\rm t} q^S_{nl}}{C_{\rm l} k^S_{nl}} \,, \qquad D_{nl} \equiv \frac{q^S_{nl}}{k^S_{nl}} E \,, \qquad \tilde{D}_{nl} \equiv \frac{C_{\rm t} F q^S_{nl}}{C_{\rm l} k^S_{nl}} \tag{4.61}$$

and introduced the normalisation constant C_{nl}, which is fixed by the orthogonality properties

$$\int_V (\mathbf{u}^S_{n'l'm'})^* \cdot (\mathbf{u}^S_{nlm})\, \varrho_0\, d^3x = M\, \delta_{nn'} \delta_{ll'} \delta_{mm'} \tag{4.62}$$

where M is the mass of the hollow sphere:

$$M = \frac{4\pi}{3}\,\varrho_0 R^3\,(1-\varsigma^3)\ , \qquad \varsigma \equiv \frac{a}{R} \leq 1 \tag{4.63}$$

Equation (4.62) fixes the value of C_{nl} through the radial integral

$$\int_{\varsigma R}^{R} \left[A_{nl}^2(r) + l(l+1)\,B_{nl}^2(r)\right]\,r^2dr = \frac{4\pi}{3}\varrho_0\,(1-\varsigma^3)R^3 \tag{4.64}$$

as can be easily verified by using well known properties of angular momentum operators and spherical harmonics. I shall later specify the values of the different parameters appearing in the above expressions as required in each particular case which will in due course be considered. As seen in reference [9], a scalar–tensor theory of GWs such as JBD predicts the excitation of the sphere's monopole modes *as well as the* $m=0$ quadrupole modes. In order to calculate the energy absorbed by the detector according to that theory it is necessary to calculate the energy deposited by the wave in those modes, and this in turn requires that I solve the elasticity equation with the GW driving term included in its right hand side. The result of such calculation was presented in full generality in reference [9], and is directly applicable here because the structure of the oscillation eigenmodes of a hollow sphere is equal to that of the massive sphere —only the explicit form of the wavefunctions needs to be changed. I thus have

$$E_{\rm osc}(\omega_{nl}) = \frac{1}{2}\,Mb_{nl}^2\sum_{m=-l}^{l}|G^{(lm)}(\omega_{nl})|^2 \tag{4.65}$$

where $G^{(lm)}(\omega_{nl})$ is the Fourier amplitude of the corresponding incoming GW mode, and

$$b_{n0} \;=\; -\frac{\varrho_0}{M}\int_a^R A_{n0}(r)\,r^3dr \tag{4.66}$$

$$b_{n2} \;=\; -\frac{\varrho_0}{M}\int_a^R \left[A_{n2}(r)+3B_{n2}(r)\right]\,r^3dr \tag{4.67}$$

for monopole and quadrupole modes, respectively, and $A_{nl}(r)$ and $B_{nl}(r)$ are given by (4.59). Explicit calculation yields

$$\frac{b_{n0}}{R} \;=\; \frac{3}{4\pi}\,\frac{C_{n0}}{1-\varsigma^3}\,\left[\Lambda(R)-\varsigma^3\Lambda(a)\right] \tag{4.68}$$

$$\frac{b_{n2}}{R} \;=\; \frac{3}{4\pi}\,\frac{C_{n2}}{1-\varsigma^3}\,\left[\Sigma(R)-\varsigma^3\Sigma(a)\right] \tag{4.69}$$

with

$$\Lambda(z) \;\equiv\; \frac{j_2(q_{n0}z)}{q_{n0}R} + D_{n0}\,\frac{y_2(q_{n0}z)}{q_{n0}R} \tag{4.70}$$

$$\Sigma(z) \;\equiv\; \frac{j_2(q_{n2}z)}{q_{n2}R} - 3K_{n2}\,\frac{j_2(k_{n2}z)}{k_{n2}R} + D_{n2}\,\frac{y_2(q_{n2}z)}{q_{n2}R} - 3\tilde{D}_{n2}\,\frac{y_2(k_{n2}z)}{k_{n2}R} \tag{4.71}$$

The absorption *cross section*, defined as the ratio of the absorbed energy to the incoming flux, can be calculated thanks to an *optical theorem*, as proved e.g. by Weinberg

[28]. According to that theorem, the absorption cross section for a signal of frequency ω close to ω_N, say, the frequency of the detector mode excited by the incoming GW, is given by the expression

$$\sigma(\omega) = \frac{10\,\pi\eta c^2}{\omega^2}\,\frac{\Gamma^2/4}{(\omega-\omega_N)^2+\Gamma^2/4} \tag{4.72}$$

where Γ is the *linewitdh* of the mode —which can be arbitrarily small, as assumed in the previous section—, and η is the dimensionless ratio

$$\eta = \frac{\Gamma_{\rm grav}}{\Gamma} = \frac{1}{\Gamma}\,\frac{P_{GW}}{E_{\rm osc}} \tag{4.73}$$

where P_{GW} is the energy *re-emitted* by the detector in the form of GWs as a consequence of its being set to oscillate by the incoming signal. In the following I will only consider the case $P_{GW} = P_{\rm scalar-tensor}$ with [9, 6]

$$P_{\rm scalar-tensor} = \frac{2G\,\omega^6}{5c^5\,(2\omega_{BD}+3)}\,\left[\left|Q_{kk}(\omega)\right|^2 + \frac{1}{3}\,Q^*_{ij}(\omega)Q_{ij}(\omega)\right] \tag{4.74}$$

where $Q_{ij}(\omega)$ is the quadrupole moment of the hollow sphere:

$$Q_{ij}(\omega) = \int_{\rm Antenna} x_i x_j\,\varrho({\bf x},\omega)\,d^3x \tag{4.75}$$

and ω_{BD} is Brans–Dicke's parameter.

5. Scalar-tensor cross sections

Explicit calculation shows that $P_{\rm scalar-tensor}$ is made up of two contributions:

$$P_{\rm scalar-tensor} = P_{00} + P_{20} \tag{5.76}$$

where P_{00} is the scalar, or monopole contribution to the emitted power, while P_{20} comes from the central quadrupole mode which, as discussed in [6] and [9], is excited together with monopole in JBD theory. One must however recall that monopole and quadrupole modes of the sphere happen at *different frequencies*, so that cross sections for them only make sense if defined separately. More precisely,

$$\sigma_{n0}(\omega) = \frac{10\pi\,\eta_{n0}\,c^2}{\omega^2}\,\frac{\Gamma^2_{n0}/4}{(\omega-\omega_{n0})^2+\Gamma^2_{n0}/4} \tag{5.77}$$

$$\sigma_{n2}(\omega) = \frac{10\pi\,\eta_{n2}\,c^2}{\omega^2}\,\frac{\Gamma^2_{n2}/4}{(\omega-\omega_{n2})^2+\Gamma^2_{n2}/4} \tag{5.78}$$

where η_{n0} and η_{n2} are defined like in (4.73), with all terms referring to the corresponding modes. After some algebra one finds that

$$\sigma_{n0}(\omega) = H_n\,\frac{GMv_S^2}{(\omega_{BD}+2)\,c^3}\,\frac{\Gamma^2_{n0}/4}{(\omega-\omega_{n0})^2+\Gamma^2_{n0}/4} \tag{5.79}$$

$$\sigma_{n2}(\omega) = F_n\,\frac{GMv_S^2}{(\omega_{BD}+2)\,c^3}\,\frac{\Gamma^2_{n2}/4}{(\omega-\omega_{n2})^2+\Gamma^2_{n2}/4} \tag{5.80}$$

Here, I have defined the dimensionless quantities

$$H_n = \frac{4\pi^2}{9\,(1+\sigma_P)}\,(k_{n0}b_{n0})^2 \tag{5.81}$$

$$F_n = \frac{8\pi^2}{15\,(1+\sigma_P)}\,(k_{n2}b_{n2})^2 \tag{5.82}$$

where σ_P represents the sphere material's Poisson ratio (most often very close to a value of 1/3), and the b_{nl} are defined in (4.68); v_S is the speed of sound in the material of the sphere.

In tables 1 and 2 I give a few numerical values of the above cross section coefficients.

Table 1. Eigenvalues $k^S_{n0}R$, relative weights D_{n0} and H_n coefficients for a hollow sphere with Poisson ratio σ_P=1/3. Values are given for a few different thickness parameters ς.

ς	n	$k^S_{n0}R$	D_{n0}	H_n
0.01	1	5.48738	-.000143328	0.90929
	1	12.2332	-.001.59636	0.14194
	2	18.6321	-.00558961	0.05926
	4	24.9693	-.001279	0.03267
0.10	1	5.45410	-0.014218	0.89530
	1	11.9241	-0.151377	0.15048
	2	17.7277	-0.479543	0.04922
	4	23.5416	-0.859885	0.04311
0.15	1	5.37709	-0.045574	0.86076
	2	11.3879	-0.434591	0.17646
	3	17.105	-0.939629	0.05674
	4	23.605	-0.806574	0.05396
0.25	1	5.04842	-0.179999	0.73727
	2	10.6515	-0.960417	0.30532
	3	17.8193	-0.425087	0.04275
	4	25.8063	0.440100	0.06347
0.50	1	3.96914	-0.631169	0.49429
	2	13.2369	0.531684	0.58140
	3	25.4531	0.245321	0.01728
	4	37.9129	0.161117	0.07192
0.75	1	3.26524	-0.901244	0.43070
	2	25.3468	0.188845	0.66284
	3	50.3718	0.093173	0.00341
	4	75.469	0.061981	0.07480
0.90	1	2.98141	-0.963552	0.42043
	2	62.9027	0.067342	0.67689
	3	125.699	0.033573	0.00047
	4	188.519	0.022334	0.07538

As already stressed in reference [27], one of the main advantages of a hollow sphere is that it enables to reach good sensitivities at lower frequencies than a solid sphere.

Table 2. Eigenvalues $k^S_{n2}R$, relative weights K_{n2}, D_{n2}, $\tilde{D}_{n2}$ and F_n coefficients for a hollow sphere with Poisson ratio $\sigma_P = 1/3$. Values are given for a few different thickness parameters ς.

ς	n	$k^S_{n2}R$	K_{n2}	D_{n2}	$\tilde{D}_{n2}$	F_n
0.10	1	2.63836	0.855799	0.000395	-0.003142	2.94602
	2	5.07358	0.751837	0.002351	-0.018451	1.16934
	3	10.96090	0.476073	0.009821	-0.071685	0.02207
0.15	1	2.61161	0.796019	0.001174	-0.009288	2.86913
	2	5.02815	0.723984	0.007028	-0.053849	1.24153
	3	8.25809	-2.010150	-0.094986	0.672786	0.08113
0.25	1	2.49122	0.606536	0.003210	-0.02494	2.55218
	2	4.91223	0.647204	0.019483	-0.13867	1.55022
	3	8.24282	-1.984426	-0.126671	0.67506	0.05325
	4	10.97725	0.432548	-0.012194	0.02236	0.03503
0.50	1	1.94340	0.300212	0.003041	-0.02268	1.61978
	2	5.06453	0.745258	0.005133	-0.02889	2.29572
	3	10.11189	1.795862	-1.697480	2.98276	0.19707
	4	15.91970	-1.632550	-1.965780	-0.30953	0.17108
0.75	1	1.44965	0.225040	0.001376	-0.01017	1.15291
	2	5.21599	0.910998	-0.197532	0.40944	1.82276
	3	13.93290	0.243382	0.748219	-3.20130	1.08952
	4	23.76319	0.550278	-0.230203	-0.81767	0.08114
0.90	1	1.26565	0.213082	0.001019	-0.00755	1.03864
	2	4.97703	0.939420	-0.323067	0.52279	1.54106
	3	31.86429	6.012680	-0.259533	4.05274	1.46486
	4	61.29948	0.205362	-0.673148	-1.04369	0.13470

For example, a hollow sphere of the same material and mass as a solid one ($\varsigma = 0$) has eigenfrequencies which are smaller by

$$\omega_{nl}(\varsigma) = \omega_{nl}(\varsigma = 0)\,(1 - \varsigma^3)^{1/3} \tag{5.83}$$

for any mode indices n and l. I now consider the detectability of JBD GW waves coming from several interesting sources with a hollow sphere.

The values of the coefficients F_n and H_n, together with the expressions (5.77) for the cross sections of the hollow sphere, can be used to estimate the maximum distances at which a coalescing compact binary system and a gravitational collapse event can be seen with such detector. I consider these in turn.

By taking as a source of GWs a binary system formed by two neutron stars, each of them with a mass of $m_1 = m_2 = 1.4\,M_\odot$. The *chirp mass* corresponding to this system is $M_c \equiv (m_1 m_2)^{3/5}\,(m_1 + m_2)^{-1/5} = 1.22\,M_\odot$, and $\nu_{[5\ \mathrm{cycles}]} = 1270$ Hz. Repeating the analysis carried on in Section three I find a formula for the minimum distance at which a measurement can be performed given a certain signal to noise ratio (SNR), for a *quantum limited* detector

$$r(\omega_{n0}) = \left[\frac{5\cdot 2^{1/3}}{32}\frac{1}{(\Omega_{BD}+2)(12\Omega_{BD}+19)}\frac{G^{5/3}M_c^{5/3}}{c^3}\frac{Mv_S^2}{\hbar\omega_{n0}^{4/3}SNR}H_n\right]^{1/2} \quad (5.84)$$

$$r(\omega_{n2}) = \left[\frac{5\cdot 2^{1/3}}{192}\frac{1}{(\Omega_{BD}+2)(12\Omega_{BD}+19)}\frac{G^{5/3}M_c^{5/3}}{c^3}\frac{Mv_S^2}{\hbar\omega_{n2}^{4/3}SNR}F_n\right]^{1/2} \quad (5.85)$$

For a CuAL sphere, the speed of sound is $v_S = 4700$ m/sec. I report in table 3 the maximum distances at which a JBD binary can be seen with a 100 ton hollow spherical detector, including the size of the sphere (diameter and thickness factor) for $SNR = 1$. The Brans-Dicke parameter Ω_{BD} has been given a value of 600. This high value has as a consequence that only relatively nearby binaries can be scrutinised by means of their scalar radiation of GWs. A slight improvement in sensitivity is appreciated as the diameter increases in a fixed mass detector. Vacancies in the tables mean the corresponding frequencies are higher than $\nu_{[5\text{ cycles}]}$.

Table 3. Eigenfrequencies, sizes and distances at which coalescing binaries can be seen by monitoring of their emitted JBD GWs. Figures correspond to a 100 ton CuAl hollow sphere.

ς	Φ (m)	ν_{10}(Hz)	ν_{12} (Hz)	$r(\nu_{10})$ (kpc)	$r(\nu_{12})$ (kpc)
0.00	2.94	1655	807	–	29.8
0.25	2.96	1562	771	–	30.3
0.50	3.08	1180	578	55	31.1
0.75	3.5	845	375	64	33
0.90	4.5	600	254	80	40

The signal associated to a gravitational collapse can be modeled, within JBD theory, as a short pulse of amplitude b, whose value can be estimated as[29]

$$b \simeq 10^{-23}\left(\frac{500}{\Omega_{BD}}\right)\left(\frac{M_*}{M_\odot}\right)\left(\frac{10\,Mpc}{r}\right) \quad (5.86)$$

Table 4. Eigenfrequencies, sizes and distances at which coalescing binaries can be seen by monitoring of their emitted JBD GWs. Figures correspond to a 3 metres external diameter CuAl hollow sphere.

ς	M (ton)	ν_{10}(Hz)	ν_{12}(Hz)	$r(\nu_{10})$ (kpc)	$r(\nu_{12})$ (kpc)
0.00	105	1653	804	–	33
0.25	103.4	1541	760	–	31
0.50	92	1212	593	52	27.6
0.75	60.7	997	442	44.8	23
0.90	28.4	910	386	32	16.3

where M_* is the collapsing mass.

The minimum value of the Fourier transform of the amplitude of the scalar wave, for a quantum limited detector at unit signal-to-noise ratio, is given by

$$|b(\omega_{nl})|_{\rm min} = \left(\frac{4\hbar}{Mv_S^2\omega_{nl}K_n}\right)^{1/2} \tag{5.87}$$

where $K_n = 2H_n$ for the mode with $l = 0$ and $K_n = F_n/3$ for the mode with $l = 2, m = 0$.

The duration of the impulse, $\tau \approx 1/f_c$, is much shorter than the decay time of the nl mode, so that the relationship between b and $b(\omega_{nl})$ is

$$b \approx |b(\omega_{nl})|\, f_c \qquad \text{atfrequency} \quad \omega_{nl} = 2\pi f_c \tag{5.88}$$

so that the minimum scalar wave amplitude detectable is

$$|b|_{\rm min} \approx \left(\frac{4\hbar\omega_{nl}}{\pi^2 M v_S^2 K_n}\right)^{1/2} \tag{5.89}$$

Let us now consider a hollow sphere made of molibdenum, for which the speed of sound is as high as $v_S = 5600$ m/sec. For a given detector mass and diameter, equation (5.89) tells us which is the minimum signal detectable with such detector. For example, a solid sphere of $M = 31$ tons and 1.8 metres in diameter, is sensitive down to $b_{\rm min} = 1.5 \cdot 10^{-22}$. Equation (5.86) can then be inverted to find which is the maximum distance at which the source can be identified by the scalar waves it emits. Taking a reasonable value of $\Omega_{BD} = 600$, one finds that $r(\nu_{10}) \approx 0.6$ Mpc.

Like before, I report in tables 4, 5 and 6 the sensitivities of the detector and consequent maximum distance at which the source appears visible to the device for various values of the thickness parameter ς. In table 5 a detector of mass of 31 tons has been assumed for all thicknesses, and in tables 4, 6 a constant outer diameter of 3 and 1.8 metres has been assumed in all cases.

Table 5. Eigenfrequencies, maximum sensitivities and distances at which a gravitational collapse can be seen by monitoring the scalar GWs it emits. Figures correspond to a 31 ton Mb hollow sphere.

ς	ϕ (m)	ν_{10} (Hz)	$\lvert b\rvert_{\rm min}$ (10^{-22})	$r(\nu_{10})$ (Mpc)
0.00	1.80	3338	1.5	0.6
0.25	1.82	3027	1.65	0.5
0.50	1.88	2304	1.79	0.46
0.75	2.16	1650	1.63	0.51
0.90	2.78	1170	1.39	0.6

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Table 6. Eigenfrequencies, maximum sensitivities and distances at which a gravitational collapse can be seen by monitoring the scalar GWs it emits. Figures correspond to a 1.8 metres outer diameter Mb hollow sphere.

ς	M (ton)	ν_{10} (Hz)	$\lvert b\rvert_{\min}$ (10^{-22})	$r(\nu_{10})$ (Mpc)
0.00	31.0	3338	1.5	0.6
0.25	30.52	3062	1.71	0.48
0.50	27.12	2407	1.95	0.42
0.75	17.92	1980	2.34	0.36
0.90	8.4	1808	3.31	0.24

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